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# Fermion masses and mixings in a flavour symmetric GUT

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## Abstract

We describe a supersymmetric Grand Unified Theory based on the gauge group  $SU(5)^3$ , or  $SO(10)^3$ , invariant under the interchange of any  $SU(5)$ , or  $SO(10)$ , with each family multiplet transforming non trivially under one different individual group factor. A realistic pattern of fermion masses and mixings is obtained as a result of an appropriate choice out of the many possible discretely degenerate vacua of the theory.

In the  $SO(10)$  case, we predict the three neutrino masses in terms of an overall scale and, within factors of order unity, their mixing angles. A  $\nu_e$ - $\nu_\mu$  oscillation is suggested as a solution of the solar neutrino problem, implying a visible  $\nu_\mu$ - $\nu_\tau$  oscillation in the forthcoming experiments.

Grand unified theories of this type could be obtainable in a string theory framework.

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# 1 Introduction

Finding a rationale in the observed pattern of fermion masses and mixings is one of the most challenging problems in today particle physics. Certainly in the Standard Model, but also in current GUTs, it is in describing the flavour sector that most of the elegance of the gauge theory structure gets lost in a plethora of arbitrary parameters. Perhaps most annoying than everything else is the fact that the clear hierarchical pattern of fermion masses is accounted for by a progression of smaller and smaller Yukawa couplings. In the language of symmetries, when the Yukawa couplings are switched off, the Standard Model lagrangian has a huge unobserved global symmetry which must be broken in a hierarchical way. Current GUTs, by extending the gauge group, do reduce such a global symmetry but always leave intact the part of it which is related to the family replicas. The fate of this residual symmetry is, to a large extent, the hearth of the flavour problem.

In this work we propose a supersymmetric GUT based on the gauge group  $SU(5)^3$ , or  $SO(10)^3$ , which leaves no residual non-abelian flavour symmetry and where, in fact, flavour is treated in a completely symmetric way. The triplication of the  $SU(5)$  ( $SO(10)$ ) factors of the overall gauge group is in one-to-one correspondence with the three family replicas. A realistic pattern of fermion masses and mixings is obtained as a result of an appropriate choice out of the many possible discretely degenerate vacua of the theory. First of all, such a choice determines the lightness of a pair of Higgs doublets, which are split from the coloured triplet partners. In turn, this is where the seed of the hierarchy in the fermion masses resides. The light Higgs doublets do transform only under one of the individual  $SU(5)$  factors of the gauge group. As a consequence, only the fermions of one family have a sizeable coupling to the light Higgs bosons, whereas the lighter families get coupled to them by a stepwise procedure, again determined by a specific vacuum structure. As a result, it is possible to describe the quark and lepton masses and mixings in terms of two/three appropriately chosen small parameters, a part from dimensionless couplings of order unity. Along these lines, by going to  $SO(10)$ , we predict the three neutrino masses in term of an overall scale and, within factors of order unity, their mixing angles. A  $\nu_e$ - $\nu_\mu$  oscillation is suggested as a solution of the solar neutrino problem, implying a visible  $\nu_\mu$ - $\nu_\tau$  oscillation in the forthcoming experiments at CERN.

As an independent motivation for such kind of theories, we are lead to consider models based on gauge groups made of several identical group factors (two or more) by the considerations of unified theories whose symmetry can be reduced to the standard  $SU(3) \otimes SU(2) \otimes U(1)$  by Higgs fields in the fundamental representation [1]. There are reasons to think that such theories can be obtained in a string theory framework [2]. Since the triplication of the gauge group factor is related to the number of families, it appears that  $SO(10)^3$  is the largest possible gauge group that one can obtain starting from the  $SO(32)$  symmetry of string theory [3].

## 2 Symmetries and multiplet structure

The gauge group is

$$SU(5)_1 \otimes SU(5)_2 \otimes SU(5)_3 = \bigotimes_{i=1}^3 SU(5)_i \quad (1)$$

The index  $i$  is associated with the family index in the sense that the  $i$ -th family multiplet

$$f_i = (\bar{5} \oplus 10)_i, \quad i = 1, 2, 3 \quad (2)$$

only transforms under the  $i$ -th factor of the gauge group<sup>1</sup>. We require invariance of the theory under the permutations  $\mathcal{P}_{ij}$  of any pair of two indices  $i, j = 1, 2, 3$ .

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<sup>1</sup>Different family multiplets transforming under different gauge group factors have been advocated by Froggatt, Lowe and Nielsen [4]. Our approach differs from theirs both because we work in a unified context and because we require permutation invariance between different families.

The minimal set of fields needed to break the gauge group down to standard  $SU(3) \otimes SU(2) \otimes U(1)$  consistently with permutation symmetry is given by the triplet

$$Z_{a_k}^{a_i} \equiv Z_{ik}, \quad \bar{Z}_{a_i}^{a_k} \equiv \bar{Z}_{ik}, \quad i < k; \quad i, k = 1, 2, 3 \quad (3)$$

where the lower (upper) indices  $a_i$  are in the (anti-)fundamental representation of the  $i$ -th  $SU(5)$ . With the definition

$$\bar{Z}_{ik} \equiv Z_{ki}$$

the permutation symmetry acts on the  $Z$ -fields by simple interchange of the corresponding indices. Finally the light Higgs doublets are contained in

$$H_{a_i} \equiv H_i, \quad \bar{H}^{a_i} \equiv \bar{H}_i, \quad i = 1, 2, 3, \quad (4)$$

with an obvious action on them of  $\mathcal{P}_{ij}$ .

### 3 Higgs superpotential

Let us define (recalling that  $i < k$ )

$$\begin{aligned} Z_{ik} &\equiv Z_j & j \neq i, j \neq k \\ \bar{Z}_{ik} &\equiv \bar{Z}_j & j \neq i, j \neq k \end{aligned} \quad (5)$$

The most general superpotential,  $W(Z)$ , dependent upon the  $Z$ -fields, up to quartic terms has the form

$$\begin{aligned} W(Z) = & M_1 \sum_j (Z_j \bar{Z}_j) + \frac{1}{M_2} \sum_{j \neq k} (Z_j \bar{Z}_j)(Z_k \bar{Z}_k) + \frac{1}{M_3} \sum_j (Z_j \bar{Z}_j)^2 + \\ & + \frac{1}{M_4} \sum_{j \neq k} (Z_j \bar{Z}_j Z_k \bar{Z}_k) + \frac{1}{M_5} \sum_j (Z_j \bar{Z}_j Z_j \bar{Z}_j) \end{aligned} \quad (6)$$

where the parentheses denote traces over the gauge group indices. For simplicity we impose invariance of  $W(Z)$  under  $Z \rightarrow -Z$ , even though the inclusion of cubic terms would not alter the final result, and we do not consider higher order terms. We look for a supersymmetric minimum of the potential of the form

$$(Z_j)_b^a = z_{ja} \delta_b^a, \quad (\bar{Z}_j)_b^a = \bar{z}_{ja} \delta_b^a \quad (7)$$

The condition of vanishing of the  $F$ -terms

$$\frac{\partial W}{\partial Z_j} = \frac{\partial W}{\partial \bar{Z}_j} = 0 \quad (8)$$

as well as of the  $D$ -terms gives rise to a large number of discretely degenerate minima. Of physical interest are the vacuum configurations of the form, for a given  $j$ ,

$$\langle Z_j \rangle = \langle \bar{Z}_j \rangle = T_j \cdot \text{diag}(1, 1, 1, 0, 0) \quad (9a)$$

$$\langle Z_j \rangle = \langle \bar{Z}_j \rangle = D_j \cdot \text{diag}(0, 0, 0, 1, 1) \quad (9b)$$

$$\langle Z_j \rangle = \langle \bar{Z}_j \rangle = V_j \cdot \text{diag}(1, 1, 1, a, a), \quad a \neq 0. \quad (9c)$$

The main constraint, in a given triplet representation of the permutation group, is that, if two or more fields have the same configuration, then the modulus of their vacuum expectation values coincide. It is only with the inclusion of higher order terms in the superpotential (6) that these vacuum expectation values can, but need not, be split.

Examples of minima which give rise to the desired breaking of the gauge group down to the standard  $SU(3) \otimes SU(2) \otimes U(1)$  are (with a choice of the permutation indices),

$$Z_{12} = Z_{13} = V \cdot \text{diag}(1, 1, 1, a, a), \quad Z_{23} = T \cdot \text{diag}(1, 1, 1, 0, 0), \quad (10)$$

or

$$Z_{12} = V \cdot \text{diag}(1, 1, 1, a, a), \quad Z_{13} = D \cdot \text{diag}(0, 0, 0, 1, 1), \quad Z_{23} = T \cdot \text{diag}(1, 1, 1, 0, 0). \quad (11)$$

Around these minima, no state remains light other than the eaten up Goldstone bosons.

Out of the  $H$ -fields, one wants a pair of  $SU(2)$  doublets to remain light. This is achieved by imposing a  $Z_2$  symmetry on the  $H$ -fields and on the  $Z$ -triplet of fields that are coupled to them in the superpotential, under which

$$\{\bar{H}, Z, \bar{Z}\} \rightarrow -\{\bar{H}, Z, \bar{Z}\} \quad (12)$$

whereas  $H$  stays invariant. At renormalizable level, the superpotential coupling the  $H$  with the  $Z$ -fields has the form

$$W(H, Z) = \lambda \sum_{i,k} H_i Z_{ik} \bar{H}_k, \quad (13)$$

or, more explicitly,

$$W(H, Z) = \lambda \begin{pmatrix} \bar{H}_1, \bar{H}_2, \bar{H}_3 \end{pmatrix} \begin{pmatrix} 0 & Z_{12} & Z_{13} \\ \bar{Z}_{12} & 0 & Z_{23} \\ \bar{Z}_{13} & \bar{Z}_{23} & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}. \quad (14)$$

As a possible example, if the  $Z$ -triplet entering into  $W(Z)$ , eq. (6), and  $W(H, Z)$ , eq. (13), coincide, in the vacuum (10) all the  $H$ -multiplets get a mass except a pair of  $SU(2)$ -doublets. Even with the inclusion in the superpotential of higher dimensional terms, these doublets remain light provided, the operators of factorized form

$$(H Z^n \bar{H}) \text{Tr}(Z^{m+1}), \quad n + m = 2k, \quad k = 1, 2, \dots, \quad (15)$$

are sufficiently suppressed. In this vacuum configuration, with  $V$  and  $T$  both of order  $10^{16}$  GeV, the observed strong and electroweak coupling constants unify at a common value  $g/\sqrt{3}$ , where  $g$  is the original gauge coupling.

## 4 Yukawa superpotential and fermion mass matrices. General expressions

The Yukawa superpotential has the following generic form

$$W_Y = \lambda^u \sum_i H_i 10_i 10_i + \lambda^d \sum_i \bar{H}_i 10_i \bar{5}_i + \sum_{n,m} \frac{\lambda_{nm}^u}{M_u^{n+m}} H Z^n 10 Z^m 10 + \sum_{n',m'} \frac{\lambda_{nm}^d}{M_d^{n+m}} \bar{H} Z^{n'} 10 Z^{m'} \bar{5} \quad (16)$$

where the higher dimensional operators have been written in schematic form. We have in mind the possibility that the higher dimensional operators in (16) be generated by exchanges of heavy vectorlike states, which might give rise to different energy scales relevant to the  $H 10 10$  and to the  $\bar{H} 10 \bar{5}$  operators,  $M_u$  and  $M_d$  respectively. Operators possibly contributing to neutrino masses will be discussed later on, in a more interesting  $SO(10)$  context.

If a unique  $Z$ -triplet enters  $W(Z)$  and  $W(H, Z)$ , and if we take (10) as the relevant vacuum configuration, the light Higgs doublets would be found in the linear combinations  $(H_2 - H_3)/\sqrt{2}$  and  $(\bar{H}_2 - \bar{H}_3)/\sqrt{2}$ . As a consequence, from the renormalizable terms in  $W_Y$ , one family remains massless whereas the two others are degenerate in the 10 and in the  $\bar{5}$  sectors separately. Assuming a small perturbation from higher order terms, although we do see a possible connection between the lightness of the Higgs doublets and the hierarchical structure of the fermion masses, this is not a reasonable starting point. To cure it, one can either assume a significant splitting between the lower components of  $Z_{12}$  and  $Z_{13}$  in the vacuum (10), or take a different  $Z$ -triplet entering  $W(H, Z)$  with only one component, say  $Z_{12}$ , having a non vanishing vacuum expectation value in the two lower entries. In this last case, to which we stick, the light doublets reside in  $H_3, \bar{H}_3$ .

From  $W_Y$ , the insertion of the  $Z$ -vacuum expectation value in the upper three or lower two components, called  $T$  and  $D$  respectively hereafter, generates the effective Yukawa couplings to the light doublets. The higher dimensional operators play an essential role both in generating the hierarchy of the diagonal couplings, since the light doublets only reside in the  $H_3$  ( $\bar{H}_3$ ), and in generating the off-diagonal entries. At the  $SU(3) \otimes SU(2) \otimes U(1)$  level, the structure of the higher order operators for the up and down mass matrices are the following

$$H D^n Q T^{m_u} c, \quad \bar{H} D^{n'} Q T^{m'_d} c \quad (17)$$

whereas, for the leptons, one has

$$\bar{H} D^{n''} L D^{m''} e^c \quad (18)$$

$T$ 's are essential to give rise to off-diagonal quark operators, whereas for leptons off-diagonal operators are only generated by vacuum expectation values in the  $D$ -direction. The different fermion mass matrices,

$$Q_i M_{ij}^u u_j^c, \quad Q_i M_{ij}^d d_j^c, \quad L_i M_{ij}^e e_j^c, \quad (19)$$

by making an expansion in  $T/M_{u,d}$ ,  $D/M_{u,d}$ , have therefore the following general form

$$M_{ij}^{u,d,e} = \mu_i^{u,d,e} V_{ij}^{u,d,e} \quad (20)$$

where (with the usual meaning of the angle  $\beta$ ),

$$\mu_i^u = v \sin \beta \left( \lambda^u \delta_{3i} + \frac{D_{3i}}{M_u} + \sum_j \frac{D_{3j} D_{ji}}{M_u^2} + \dots \right) \quad (21)$$

$$\mu_i^{d,e} = v \cos \beta \left( \lambda^d \delta_{3i} + \frac{D_{3i}}{M_d} + \sum_j \frac{D_{3j} D_{ji}}{M_d^2} + \dots \right), \quad (22)$$

and

$$V_{ij}^u = \delta_{ij} + \frac{(T_{ij})^2}{M_u^2} + \dots \quad (23a)$$

$$V_{ij}^d = \delta_{ij} + \frac{T_{ij}}{M_d} + \sum_k \frac{T_{ik} T_{kj}}{M_d^2} + \dots \quad (23b)$$

$$V_{ij}^e = \delta_{ij} + \frac{D_{ij}}{M_d} + \sum_k \frac{D_{ik} D_{kj}}{M_d^2} + \dots \quad (23c)$$

In front of every mass term involving a  $D$  or a  $T$ , a dimensionless coupling has been left understood, in general also different for leptons and down quarks. Possible relations between the corresponding entries in the  $d$  and the  $\ell$ -mass matrices may in fact occur, as a remnant of the SU(5) symmetry, depending on the more specific structure of the theory (see below). The difference in the SU(5) representations that contain the  $u^c$  (10) and the  $d^c$  ( $\bar{5}$ ) is the source of the difference in the corresponding off-diagonal terms in  $V_{ij}^u$  and  $V_{ij}^d$ .

It is manifest from the structure of the mass matrices (19–23) that  $D$  and  $T$ -vacuum expectation values smaller than  $M_{u,d}$  can induce a hierarchy in the masses and in the mixing angles of the fermions, in spite of the original flavour symmetry. As we shall see, this possibility has everything to do with the huge number of different vacua present in the theory. The degeneracy of those vacua will be lifted as a consequence of supersymmetry breaking and the vacuum of interest will be converted into a global or a local minimum depending upon the specific form of the potential and its parameters.

At the same time it is important to observe, as a neat property of the theory that we are discussing, that the tree level soft supersymmetry breaking masses of the squarks (and of the leptons) are necessarily flavour degenerate if supersymmetry breaking, as expected, preserves the permutation symmetry.

## 5 Fermion masses matrices: towards a realistic structure

Other than its interesting features, however, the general kind of theories that have been discussed so far have two quite manifest problems. The first one is the lack of sufficient asymmetry between the up and down quark mass matrices, which is especially embarrassing in view of the large top-bottom mass difference. The second one is related to the proton decay amplitude mediated by triplet exchanges. In fact, we generally expect such amplitude to be significantly larger than in a standard “minimal” SU(5) theory, since at least one of the 3 coloured triplets in  $H_i$  will have large couplings, of order unity, to the first generation as a consequence of the permutation symmetry. Even if the mass of such triplet were pushed to the Planck scale, the corresponding amplitude could not be sufficiently suppressed.

We think that the only common solution of both problems resides in barring the renormalizable coupling of the down-type quarks in the Yukawa superpotential, as it can be enforced by an appropriate set of  $\mathcal{Z}_2$ -symmetries, which extend the one defined in eq. (12). In particular, we propose the following structure as a possible example of a realistic theory of the mass matrices.

The  $H$ -fields occur in two replicas

$$H_i, \bar{H}_i; \quad H'_i, \bar{H}'_i, \quad (24)$$

whereas the  $Z$ -fields occur in several replicas, which must include the fields

$$Z, Z', Y, Y'. \quad (25)$$

On these fields it is simple to write down a set of  $\mathcal{Z}_2$ -symmetries which enforce the following structure of the Higgs and Yukawa superpotential (omitting dimensionless couplings and summations over permutation indices)

$$W(H, Z) = HZ\bar{H} + H'Z'\bar{H}' \quad (26)$$

$$W_Y = H \begin{smallmatrix} 10 & 10 \end{smallmatrix} + H Y^n \begin{smallmatrix} 10 & Y^m & 10 \end{smallmatrix} + \bar{H}' Y' \begin{smallmatrix} 10 & \bar{5} \end{smallmatrix} + \bar{H}' Y' Y^{n'} \begin{smallmatrix} 10 & Y^{m'} & \bar{5} \end{smallmatrix}. \quad (27)$$

With these superpotentials, out of the various possibilities, a vacuum which leads to a realistic structure of the mass matrices is the one where the fields that have a non-vanishing vacuum expectation value in the  $D$ -direction are  $Z_{12}$ ,  $Z'_{13}$ ,  $Y_{12}$ ,  $Y_{23}$  and  $Y'_{23}$ . Furthermore all these fields have non-zero vacuum expectation values in the  $T$ -direction except for the  $Y'$ . The essential properties of this choice are:

- $D_{Z_{12}} \neq 0$  only (out of the  $Z_{ik}$ 's) forces the light doublet coupled to the up-quarks to reside in the  $H_3$  direction only.
- In the same way  $D_{Z'_{13}} \neq 0$  gives the light doublet coupled to the down-quarks (and the leptons) in the  $H_2$  direction. However, since  $D_{Y'_{23}} \neq 0$  only,  $H_2$  has its main effective coupling, although suppressed by a factor

$$\varepsilon' \equiv \frac{\langle D_{Y'_{23}} \rangle}{M_d} \quad (28)$$

to the third generation.

- The vanishing of  $D_{Y_{13}}$  is the source of the hierarchy in the eigenvalues of the mass matrices, suppressed relative to the dominant third one by factors  $\varepsilon_d, \varepsilon_u, \varepsilon_d^2, \varepsilon_u^2$  where

$$\varepsilon_d \equiv \frac{\langle D_{Y_{13}} \rangle}{M_d}, \quad \varepsilon_u \equiv \frac{\langle D_{Y_{13}} \rangle}{M_u}. \quad (29)$$

- The vanishing of all the  $Y'$  fields in the  $T$ -direction is the source of the decoupling of the  $d$ -type quarks from the heavy triplets in  $\bar{H}$  [5], whose exchange would otherwise lead to the proton decay at a dangerous level. This last property requires some suppression of the operators  $(\bar{H}' \begin{smallmatrix} 10 & \bar{5} \end{smallmatrix}) \text{Tr}(Y' Y^n)$ . Notice that also these operators, as it was the case for (15) are of factorized form in the gauge group indices. On this basis, from now on, we shall assume that all factorized operators are absent or have anyhow a negligible strength. Notice also that the Higgs superpotential (26) leads, as it stands, to two pairs of light doublets, which is undesirable from the point of view of the unification of the coupling constants. A possible way out consists in introducing the coupling  $S\bar{H}H'$ , again consistently with the  $\mathcal{Z}_2$ -symmetries, with the singlet  $S$  getting a vacuum expectation value at a scale greater than or equal to the unification scale. SO(10) offers an alternative, more interesting, solution for this doubling of the light Higgs pairs (see below).

A systematic exploration of the Yukawa superpotential  $W_Y$  with the assumed properties of the relevant vacuum leads to the following structure of the mass matrices (19) (leaving understood a numerical coupling,  $\lambda_{ij}^{u,d,e}$ , of order unity in front of every entry)

$$M^u = v \sin \beta \begin{pmatrix} \varepsilon_u^2 & \varepsilon_u^2 t_u^2 & \varepsilon_u^2 t_u^2 \\ \varepsilon_u t_u^2 & \varepsilon_u & \varepsilon_u t_u^2 \\ t_u^2 & t_u^2 & 1 \end{pmatrix} \quad (30a)$$

$$M^d = v\varepsilon' \cos \beta \begin{pmatrix} \varepsilon_d^2 & \varepsilon_d^2 t_d & \varepsilon_d^2 t_d \\ \varepsilon_d t_d & \varepsilon_d & \varepsilon_d t_d \\ t_d & t_d & 1 \end{pmatrix} \quad (30b)$$

$$M^e = v\varepsilon' \cos \beta \begin{pmatrix} \varepsilon_d^2 & \varepsilon_d^2 & \varepsilon_d^2 \\ \varepsilon_d^3 & \varepsilon_d & \varepsilon_d \\ \varepsilon_d^2 & 1 & 1 \end{pmatrix} \quad (30c)$$

where

$$t_u = \frac{\langle T_Y \rangle}{M_u}, \quad t_d = \frac{\langle T_Y \rangle}{M_d}. \quad (31)$$

Some relations occur between the numerical couplings  $\lambda_{ij}^d$  and  $\lambda_{ij}^e$  in the  $d$  and  $\ell$ -mass matrices, as a consequence of the SU(5) symmetry. An equality in fact holds, but only for the 33 and 11 entries. This is because the 33 entries only arise from the operator (with an obvious contraction of the group indices)

$$\bar{H}'_2 \bar{Y}'_{23} 10_3 \bar{5}_3, \quad (32)$$

and the 11 entries come from

$$\bar{H}'_2 \bar{Y}'_{23} Y_{23} Y_{12} 10_1 \bar{5}_1. \quad (33)$$

On the contrary, an operator which contributes at dominant level to the 22 entry of the leptons only is

$$\bar{H}'_2 10_2 Y_{23} Y'_{32} \bar{5}_2. \quad (34)$$

As a further significant asymmetry between the two mass matrices, notice also the operator

$$\bar{H}'_2 10_2 \bar{Y}'_{32} \bar{5}_3, \quad (35)$$

which contributes, at dominant level, to the 23 entry for the leptons only ( $Y'_{23}$  has no vacuum expectation value in the  $T$ -direction).

The mass matrices (30) offer the possibility of a realistic description of fermion masses and mixings in terms of the parameters  $\varepsilon_u, \varepsilon_d, t_d$  ( $t_u = t_d \varepsilon_u / \varepsilon_d$ ) and  $\varepsilon'$  (a part from couplings of order unity) which are related to a hierarchy of scales. Without attempting a fit, in view of the unknown coefficients, but rather taking for illustration

$$\varepsilon_d = \frac{1}{20}, \quad \varepsilon_u = \frac{1}{400}, \quad (36)$$

one has, within the present uncertainties and after appropriate renormalization group rescalings [6] at the unification scale

$$\begin{aligned} \frac{m_c}{m_t} &= (0.3 \div 2.0) \cdot \varepsilon_u \\ \frac{m_u}{m_c} &= (1.0 \div 2.5) \cdot \varepsilon_u \\ \frac{m_s}{m_b} &= (0.2 \div 1.0) \cdot \varepsilon_d \\ \frac{m_d}{m_s} &= (0.5 \div 2.0) \cdot \varepsilon_d \\ \frac{m_\mu}{m_\tau} &= 1.20 \cdot \varepsilon_d \\ V_{us} &= 4.41 \cdot \varepsilon_d \\ \frac{m_e}{m_\mu} &= 0.097 \cdot \varepsilon_d \end{aligned}$$

Furthermore, with  $\varepsilon' = \tan \beta / 200$  and  $t_d = 1$ ,

$$\begin{aligned} \frac{m_b}{m_t} &= (0.2 \div 5) \cdot \frac{\varepsilon'}{\tan \beta} \\ \frac{V_{cb} m_b}{m_s} &= (0.5 \div 2.5) \cdot t_d \\ \frac{V_{ub} m_s}{V_{cb} m_b} &= (0.5 \div 3.0) \cdot t_d \end{aligned}$$

The greatest uncertainty in the various coefficients is due to the poor knowledge of the top Yukawa coupling at the unification scale, which we take to vary from 0.3 to 4, so that  $M_t^{\text{pole}} = (150 \div 200)$  GeV and the perturbative expansion is maintained up to unification.

In view of the unknown coefficients of order unity that enter the matrices (30), we find an overall remarkably consistent picture of all masses and mixings. This is achieved with the choice of scales

$$M_u \approx 20 \{M_d, T_Y\} \approx 400 D_Y \approx \frac{4000}{\tan \beta} D_{Y'} \quad (37)$$

Notice that, due to the smallness of the parameter  $\varepsilon_u$ , the elements of the CKM matrix are only determined by  $M^d$ .

## 6 Extension to $\text{SO}(10)^3$

The model described in the previous sections can be extended to the gauge group  $\text{SO}(10)^3$  with two aspects deserving a special discussion: the solution of the proton decay problem (related to the up/down mass splitting) and the neutrino masses. The Higgs fields required to break the group down to  $\text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1)$  are a triplet of fields with a pair of vector indices in two of the three  $\text{SO}(10)$  factors

$$Z_{a_i a_k} \equiv Z_{ik}, \quad i < k; \quad i, k = 1, 2, 3, \quad (38)$$

and a triplet of fields transforming as the spinorial representation of the same  $\text{SO}(10)$ 's, together with the corresponding conjugate representations

$$\psi_{\alpha_i} \equiv \psi_i; \quad \bar{\psi}_{\alpha_i} \equiv \bar{\psi}_i, \quad i = 1, 2, 3. \quad (39)$$

From a generic superpotential  $W(Z)$ , the discussion of the vacuum expectation values of the fields  $Z_{ik}$  is identical to the one of section 3 for the  $\text{SU}(5)$  case. Of interest are the vacuum configurations

$$\langle Z \rangle = \langle \bar{Z} \rangle = T_j \cdot \text{diag}(1, 1, 1, 1, 1, 1, 0, 0, 0, 0) \quad (40a)$$

$$\langle Z \rangle = \langle \bar{Z} \rangle = D_j \cdot \text{diag}(0, 0, 0, 0, 0, 0, 1, 1, 1, 1) \quad (40b)$$

$$\langle Z \rangle = \langle \bar{Z} \rangle = V_j \cdot \text{diag}(1, 1, 1, 1, 1, 1, a, a, a, a), \quad a \neq 0. \quad (40c)$$

which break the  $\text{SO}(10) \otimes \text{SO}(10)$  group, under which the individual  $Z_{ik}$  transform, down to

$$\text{SO}(6) \otimes \text{SO}(4) \otimes \text{SO}(4)$$

$$\text{SO}(6) \otimes \text{SO}(6) \otimes \text{SO}(4)$$

$$\text{SO}(6) \otimes \text{SO}(4)$$

respectively.

In the same way, from a generic superpotential in the 16-plets,  $W(\psi, \bar{\psi})$ , one obtains for them a vacuum solution which preserves the relative  $\text{SU}(5)$ 's, with expectation values, for the different  $\langle \psi_i \rangle = \langle \bar{\psi}_i \rangle$ , which either vanish or coincide (for a superpotential including up to quartic terms). As in standard  $\text{SO}(10)$ , with the appropriate embedding of the  $\text{SU}(5)$ 's in the corresponding  $\text{SO}(10)$ 's, the intersection of the different subgroups leads to the residual  $\text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1)$ . Of course, to avoid uneaten Goldstone-bosons, the superpotential must also contain some interaction terms between the  $Z$  and the  $\psi, \bar{\psi}$  multiplets. The terms of lowest dimensionality are

$$\begin{aligned} W(Z, \psi, \bar{\psi}) = & \frac{1}{M^2} \sum_{i,k} Z_{a_i a_k} \{ \lambda (\psi \gamma_{a_i} \psi) (\psi \gamma_{a_k} \psi) + \\ & + \sigma (\bar{\psi} \gamma_{a_i} \bar{\psi}) (\psi \gamma_{a_k} \psi) + \tau (\bar{\psi} \gamma_{a_i} \bar{\psi}) (\bar{\psi} \gamma_{a_k} \bar{\psi}) \} \end{aligned}$$

Such terms act as connectors between the  $Z$  and the  $\psi, \bar{\psi}$  fields in the sense that they vanish on the vacuum configurations described above (the 10 of  $\text{SO}(10)$  has no singlet under  $\text{SU}(5)$ ) and they give masses to all unwanted Goldstone bosons.



In analogy with the SU(5) case, to solve the proton decay problem, the triplets of Higgs fields entering into the Yukawa superpotential must occur in two replicas of 10-plets,  $H_i, H'_i$  and be coupled to a pair of  $Z$ -type triplets

$$W(H, Z) = HZH + H'Z'H'. \quad (41)$$

With an appropriate choice of the  $Z, Z'$  vacuum expectation values, one gives masses to all coloured triplets in  $H, H'$  and remains with a pair of SU(2)-doublets in  $H_3$  ( $h_3$  and  $\bar{h}_3$ ) and a pair of doublets in  $H'_2$  ( $h'_2$  and  $\bar{h}'_2$ ). Since  $h_3$  is coupled to the up quarks (and the neutrinos), whereas  $\bar{h}'_2$  gives mass to the down quarks and the charged leptons, in order not to undo the successful unification of couplings, the two remaining doublets have to receive a heavy mass. This is achieved by the couplings, asymmetric in  $\psi, \bar{\psi}$ ,

$$W(\psi, \bar{\psi}, H) = H\bar{\psi}\psi + H'\psi\bar{\psi}. \quad (42)$$

The Yukawa superpotential with the 16-plets of matter families, always omitting dimensionless couplings and contractions over SO(10) and permutation indices, is

$$\begin{aligned} W_Y = & \sum_n \frac{1}{M_u^n} HY^n(16\ 16) + \sum_{n'} \frac{1}{M_d^{n'+1}} H'Y'Y^{n'}(16\ 16) + \\ & + \sum_{n,m} \frac{1}{M_u^{n+m+2}} HY^n(16\ \psi)Y^m(\psi\ 16) + \sum_{n',m'} \frac{1}{M_d^{n'+m'+3}} H'Y'Y^{n'}(16\ \psi)Y^{m'}(\psi\ 16) \end{aligned} \quad (43)$$

The first two terms, in conjunction with (41) and (42), give rise to the diagonal terms in  $M^u$  and  $M^d$ ,  $M^e$  respectively, whereas the last two are the source of the off diagonal matrix elements. Notice in particular in these last terms the necessity to introduce the  $\psi$ -fields.

After insertion of the various vacuum expectation values, this superpotential gives rise to the same mass matrices as in (30a,b,c), where now

$$t_d = \frac{\langle\psi\rangle^2 \langle T_Y \rangle}{M_d^3}, \quad t_u = \frac{\langle\psi\rangle^2 \langle T_Y \rangle^2}{M_u^4}, \quad (44)$$

whereas the  $\varepsilon$  parameters keep the same meaning as before. As a consequence, the description of all charged fermion masses and mixings is unchanged.

Needless to say, a special phenomenological interest of SO(10), or SO(10)<sup>3</sup>, is that it allows a discussion of neutrino masses. From the superpotential (43), the only relevant neutrino mass terms are the diagonal entries in the Dirac  $\nu\nu^c$  mass matrices, which are effectively identical to the  $Q = 2/3$  quark masses, since the off-diagonal terms are too small in both cases. Such diagonal entries arise from the first term in (43), which in any case respects the Pati-Salam SU(4).

An independent source for Dirac masses,  $\nu\nu^c$ , and only for them, is the operator

$$\frac{1}{M_\nu^{n+2}} \sum_k (16_k \bar{\psi}_k) \cdot \sum_i (HY^n)_{a_i} (16_i \gamma_{a_i} \psi_i)$$

which, depending on  $M_\nu$ , could give significant contributions to diagonal and off diagonal entries. As mentioned, we assume that such factorized operators are suppressed.

Always disregarding factorized operators, the dominant contribution to the Majorana mass term  $\nu^c \nu^c$  for the right-handed neutrinos comes from the dimension-4 superpotential term

$$\frac{1}{M_\nu} \sum_i (\bar{\psi}_i \Gamma_{a_i}^{(\overline{126})} \bar{\psi}_i) (\psi_i \Gamma_{a_i}^{(126)} \psi_i)$$

(or equivalent, by SO(10) Fiertz rearrangement), where, inside each parenthesis, the spinorial indices are contracted to make a 126 ( $\overline{126}$ ) representation of SO(10). Such term, taking  $\langle\psi_i\rangle$   $i$ -independent as explained above, gives a right handed neutrino mass matrix proportional to the identity by a factor  $\langle\psi\rangle^2/M_\nu$ . On the other hand, the dominant non factorizable contribution to a direct left-handed neutrino Majorana mass,  $\nu\nu$ , comes from an operator of very high dimension and can therefore be neglected. As a overall consequence, the effective  $\nu\nu$  mass matrix, in the original basis, is, to a good approximation, diagonal, with eigenvalues

$$m_{\nu_\tau} = \frac{m_t^2}{\langle\psi\rangle^2/M_\nu}, \quad m_{\nu_\mu} = m_{\nu_\tau} \frac{m_c^2}{m_t^2}, \quad m_{\nu_e} = m_{\nu_\mu} \frac{m_u^2}{m_c^2}, \quad (45)$$

where the  $Q = 2/3$  quark masses refer to their high scale values. Taking  $M_\nu \leq 10^{19}$  GeV and  $\langle\psi\rangle \approx 10^{16}$  GeV, we have, for  $\lambda_t^{\text{GUT}} = 0.3 \div 4$

$$m_{\nu_\tau} \leq 100 \text{ eV}, \quad m_{\nu_\mu} = (0.05 \div 3) 10^{-5} m_{\nu_\tau}, \quad m_{\nu_e} = (0.6 \div 4) 10^{-5} m_{\nu_\mu}. \quad (46)$$

As seen in section 5, in the original basis, the charged lepton mass matrix (30c) is not flavour diagonal. As a consequence, the mixing angles in the leptonic charged current weak interactions arise from the need to diagonalize  $M^e$ , so that

$$\begin{aligned} \sin 2\theta_{e\mu} &\approx \sin 2\theta_{\mu\tau} \approx 2\varepsilon_d \approx 0.1, \\ \sin 2\theta_{e\tau} &\approx 2\varepsilon_d^2 \approx 0.5 \cdot 10^{-2} \end{aligned} \quad (47)$$

It is clear from eq.s (46,47) that a  $\tau$ -neutrino saturating the bound (46) could give, depending on the value of  $\lambda_t^{\text{GUT}}$ , a consistent solution of the solar neutrino problem as a manifestation of a  $\nu_e$ - $\nu_\mu$  resonant [7] oscillation. In such a case the  $\nu_\mu$ - $\nu_\tau$  oscillation would most likely give an observable signal in the forthcoming experiments at CERN [8].

## 7 Conclusions

In conclusion, we have shown how a realistic pattern of fermion masses and mixings can be obtained in a unified supersymmetric theory where flavour is treated in a completely symmetric way. Although we do not obtain any particular relation between different masses and/or mixings angles of the charged fermions, due to the presence of unknown coefficients of order unity, we do find a simple rationale for these masses and mixings in terms of 3 large (grand unified) mass scales. Some particular relations might occur as a consequence of the choice of a more specific vacuum structure than the one that we have considered.

By going to SO(10), we predict the three neutrino masses in terms of an overall scale and, within factors of order unity, their mixing angles. A  $\nu_e$ - $\nu_\mu$  oscillation is suggested as a solution of the solar neutrino problem, implying a visible  $\nu_\mu$ - $\nu_\tau$  oscillation in the forthcoming experiments.

We have been independently laid to consider unified models of the type discussed on this paper by requiring that the unified gauge group be broken by the vacuum expectation values of Higgs fields in the fundamental representation. This criterium is suggested by string theory considerations. Models with the gauge structure and with the Higgs content suggested here seem indeed to be obtainable in a string theory context [2]. Also based on these considerations, we suggest that  $\text{SO}(10)^3$  is the largest possible ‘sensible’ subgroup of SU(45) (or rather SU(48)) that can be gauged and can be broken to the standard  $\text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1)$  group by Higgs fields in simple enough representations. At the same time, we find it particularly interesting that the residual global symmetry, in absence of Yukawa couplings, does not contain any non-abelian flavour symmetry.

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